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1980 J. Phys. A: Math. Gen. 13 1519

(http://iopscience.iop.org/0305-4470/13/5/012)

J. Phys. A: Math. Gen. 13 (1980) 1519–1536. Printed in Great Britain

Interaction of the Benjamin–Ono solitons

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Received 28 March 1979, in final form 17 September 1979

Abstract. The interaction of the Benjamin–Ono solitons is studied in great detail employing the exact N-soliton solution presented by Matsuno. For the two-soliton case, the nature of the interaction is shown to be characterised by the ratio of the amplitudes of the two solitons. Furthermore the initial-value problem of the linearised Benjamin–Ono equation is solved analytically and the asymptotic form of the solution for large time is given.

1. Introduction

In a recent study of non-linear waves, much attention has been paid to the Benjamin–Ono (B–O) equation (Benjamin 1966, 1967, Davis and Acrivos 1967, Ono 1975). The B–O equation describes a large class of internal waves which occur in the atmosphere and the ocean and it may be written as

$$\frac{\partial u(x, t)}{\partial t}+4u(x, t)\frac{\partial u(x, t)}{\partial x}+H\left[\frac{\partial^2 u(x, t)}{\partial x^2}\right]=0 \tag{1.1}$$

where $u(x, t)$ is a scalar quantity, x is the space coordinate, t is the time and H is the Hilbert transform defined by

$$H[u(x, t)]=P\int_{-\infty}^{\infty}\frac{u(y, t)}{y-x}\,\mathrm{d}y. \tag{1.2}$$

The B–O equation is a non-linear *integro-differential equation* unlike most of the well-known non-linear differential equations such as the Korteweg–deVries (KdV) equation and the non-linear Schrödinger equation. Because of the integral term (definite integral!) in the B–O equation, the inverse scattering method (Gardner *et al* 1967, 1974), which has a strong power for some class of non-linear wave equations, cannot be applied to this equation as yet.

Recently many attempts have been made to solve the B–O equation. Joseph (1977) gave a two-soliton solution. Meiss and Pereira (1978) found new conserved quantities of equation (1.1) and surmised the existence of exact two- and three-soliton solutions. Chen *et al* (1979) obtained a solution which describes multiple collisions of solitons. Matsuno (1979) presented a rational N-soliton solution explicitly applying Hirota's method (Hirota 1971); this paper (Matsuno 1979) will be referred to as I hereafter.

Very recently Case (1979) obtained a periodic solution of the B–O equation by using the pole expansion method. Satsuma and Ishimori (1979) also obtained a periodic solution by Hirota's method and showed that the solution is reduced to the rational N-soliton solution presented in I in the long-wavelength limit.

Now the stationary solution of (1.1) is given by

$$u(x, t) = \frac{V}{V^2(x - Vt - \xi)^2 + 1} \tag{1.3}$$

where V and ξ are arbitrary constants called a velocity and a phase, respectively. The solution (1.3) is remarkable because it is not a hyperbolic function like most known solutions, but instead has a Lorentzian shape.

The interaction of N solitons was treated in I and it was shown that no phase shift appears as the result of collisions between solitons, unlike those which take place between the KdV solitons (Wadati and Toda 1972, Gardner *et al* 1974). However, during the interaction itself, the process is very complicated. Therefore it seems to be interesting to study more details of the interaction process. This is the main purpose of the present paper. For the KdV equation, the interaction of two solitons has been studied in detail by some authors (Lax 1968, Kruskal 1974, Thickstun 1976). In addition, the behaviour of the tail is well known thanks to the inverse scattering method (Ablowitz and Newell 1973, Ablowitz and Segur 1977). Since this method cannot be applied to the B–O equation as yet we must resort to other methods to study the structure of the tail. As a first step to attack this problem, we solve the linearised B–O equation and study the asymptotic behaviour of the solution for large time because the amplitude of the tail is so small that it may be described by the linearised B–O equation in a first approximation. The interaction between solitons and the tail will be treated elsewhere.

In § 2, the summary of an exact method for solving the B–O equation is presented together with the rational N-soliton solution. In § 3, the interaction of two solitons is elucidated in detail using the explicit two-soliton solution. It is shown that the nature of the interaction is characterised by the ratio of the amplitudes of two solitons. In § 4, we treat the problem of the tail in the absence of solitons. Section 5 is devoted to some concluding remarks.

2. Summary of an exact method

In this section we summarise an exact method for solving the equation (1.1) (Matsuno 1979). Since we are interested in the physical solution of the equation (1.1), we seek a solution which is real and finite over all x, t and express it in the following form:

$$u(x, t) = \frac{\mathrm{i}}{2}\frac{\partial}{\partial x} \ln \frac{f^*(x, t)}{f(x, t)} \tag{2.1}$$

$$f(x, t) \propto \prod_{n=1}^{N} [x - x_n(t)] \tag{2.2}$$

$$\operatorname{Im} x_n > 0 \qquad n = 1, 2, \ldots, N \tag{2.3}$$

where $x_n (n = 1, 2, \ldots, N)$ are complex functions of t and the asterisk denotes a complex conjugate.

Substituting (2.1)–(2.3) into (1.2) and performing the contour integral closed by a large semicircle in the upper half-plane, we get

$$H[u(x,t)] = -\frac{1}{2}\left(\sum_{n=1}^{N}\frac{1}{x-x_n(t)} + \sum_{n=1}^{N}\frac{1}{x-x_n^*(t)}\right)$$
$$= -\frac{1}{2}\left(\frac{\partial f(x,t)/\partial x}{f(x,t)} + \frac{\partial f^*(x,t)/\partial x}{f^*(x,t)}\right). \tag{2.4}$$

Substituting (2.1)–(2.4) into (1.1) and integrating once with respect to x and using the boundary condition $u(x,t)\to 0$ as $|x|\to\infty$, we obtain the following bilinear equation for f:

$$\mathrm{Im}\left(\frac{\partial f^*(x,t)}{\partial t} f(x,t)\right) = \frac{\partial f^*(x,t)}{\partial x}\frac{\partial f(x,t)}{\partial x} - \mathrm{Re}\left(\frac{\partial^2 f^*(x,t)}{\partial x^2} f(x,t)\right). \tag{2.5}$$

The solution of the equation (2.5) which involves N solitons is expressed as follows (Matsuno 1979):

$$f_N = \det M \tag{2.6}$$

where M is an $N\times N$ matrix whose elements are given by

$$M_{nm} = \begin{cases} \mathrm{i}\theta_n + 1 & \text{for } n = m \\ \dfrac{2(V_n V_m)^{1/2}}{V_n - V_m} & \text{for } n \neq m \end{cases} \tag{2.7}$$

with

$$\theta_n = V_n(x - V_n t - \xi_n) \tag{2.8}$$

where V_n and $\xi_n (n = 1, 2, \ldots, N)$ are arbitrary constants and it is assumed that $V_n \neq V_m$ for $n \neq m$.

The first three of the solutions (2.6) are expressed as

$$f_1 = \mathrm{i}\theta_1 + 1 \tag{2.9a}$$

$$f_2 = -\theta_1\theta_2 + \mathrm{i}(\theta_1+\theta_2) + V_{12} \tag{2.9b}$$

$$f_3 = -\mathrm{i}\theta_1\theta_2\theta_3 - (\theta_1\theta_2 + \theta_2\theta_3 + \theta_3\theta_1)$$
$$+\mathrm{i}(V_{23}\theta_1 + V_{31}\theta_2 + V_{12}\theta_3) + V_{12} + V_{23} + V_{31} - 2 \tag{2.9c}$$

with

$$V_{nm} = [(V_n + V_m)(V_n - V_m)^{-1}]^2. \tag{2.10}$$

Note that $f_{N-1} = \lim_{V_n\to 0} f_N (n = 1, 2, \ldots, N)$. This is a remarkable property of the solution given by (2.6)–(2.8).

The two-soliton solution readily follows from (2.1) and (2.9b) as

$$u = \frac{V_2\theta_1^2 + V_1\theta_2^2 + (V_1+V_2)V_{12}}{(\theta_1\theta_2 - V_{12})^2 + (\theta_1+\theta_2)^2}. \tag{2.11}$$

This expression will be used in § 3.

At this stage it is interesting to check whether the solution (2.6) satisfies the assumption (2.3) for deriving (2.5). For this purpose we write the equation of motion of

$x_n(t)$ in the following form:

$$\frac{d}{dt}x_n(t)=2i\sum_{\substack{s=1\\(s\neq n)}}^{N}\frac{1}{x_s-x_n}+2i\sum_{s=1}^{N}\frac{1}{x_n-x_s^*}\qquad (n=1,2,\ldots,N). \tag{2.12}$$

This equation is derived easily by substituting the first expression from (2.4) into the equation (1.1) and putting the coefficients of $[x-x_n(t)]^{-2}$ zero (Chen *et al* 1979). By taking the imaginary part of the equation (2.12), we get

$$\frac{d}{dt}\,\mathrm{Im}\,x_n(t)=G_n(t)\,\mathrm{Im}\,x_n(t) \tag{2.13a}$$

where

$$G_n(t)=8\sum_{\substack{s=1\\(s\neq n)}}^{N}\frac{\mathrm{Re}(x_s-x_n)\,\mathrm{Im}\,x_s}{\{[\mathrm{Re}(x_s-x_n)]^2+[\mathrm{Im}(x_s-x_n)]^2\}\{[\mathrm{Re}(x_s-x_n)]^2+[\mathrm{Im}(x_s+x_n)]^2\}}. \tag{2.13b}$$

Integrating the equation (2.13a) with respect to t yields

$$\mathrm{Im}\,x_n(t)=\mathrm{Im}\,x_n(t_0)\exp\left[\int_{t_0}^{t}G_n(t')\,dt'\right] \tag{2.14}$$

where t_0 is an arbitrary constant which has the meaning of an initial time. It is seen from (2.14) that the conditions (2.3) are satisfied if they hold at some time t_0. In the present case it is simple to take $t_0=-\infty$. It is easy to show that (see I)

$$\mathrm{Im}\,x_n(-\infty)=1/V_n\qquad n=1,2,\ldots,N.$$

Thus, if we take $V_n>0$ $(n=1,2,\ldots,N)$, the conditions (2.3) are satisfied for the present N-soliton solution (2.6)–(2.8).

Apart from real solutions, we can find other solutions. For example, if we assume a solution in the form

$$u(x,t)=\frac{1}{2i}\frac{\partial}{\partial x}\ln f(x,t) \tag{2.15}$$

with (2.2) and (2.3), then we obtain an equation for f as (in this case N may be infinite)

$$i\frac{\partial f(x,t)}{\partial t}+\frac{\partial^2 f(x,t)}{\partial x^2}=0. \tag{2.16}$$

This equation is just a one-dimensional Schrödinger equation describing the motion of a free particle and is a *linear equation*, being different from the equation (2.5). To find the solution satisfying the conditions (2.3) will be done elsewhere.

3. Interaction of two solitons

Now let us proceed to the interaction of two solitons. To do so we rewrite the solution (2.1) in the following form:

$$u(x,t)=\frac{1}{2i}\sum_{n=1}^{N}\left(\frac{1}{x-x_n(t)}-\frac{1}{x-x_n^*(t)}\right)=\sum_{n=1}^{N}\frac{\mathrm{Im}\,x_n(t)}{[x-\mathrm{Re}\,x_n(t)]^2+[\mathrm{Im}\,x_n(t)]^2}. \tag{3.1}$$

In this expression $u(x, t)$ has $2N$ poles $x_1(t), \ldots, x_N(t), x_1^*(t), \ldots, x_N^*(t)$. The equation (3.1) indicates that the nth soliton may be characterised by the motion of the pole $x_n(t)$. The trajectory of $x_n(t)$ can be obtained directly from the exact N-soliton solution (2.6)–(2.8). It should be noted that Re x_n gives the centre position of the nth soliton and $(\mathrm{Im}\, x_n)^{-1}$ the maximum amplitude of the nth soliton.

For a two-soliton case the poles $x_1(t)$ and $x_2(t)$ are found from (2.9b) as

$$x_1(t)=\frac{V_1+V_2}{2}T+\frac{a(T)}{2}+\frac{V_1\xi_2-V_2\xi_1}{V_1V_2}+\mathrm{i}\left(\frac{V_1+V_2}{2V_1V_2}+\frac{b(T)}{2}\right) \qquad (3.2a)$$

$$x_2(t)=\frac{V_1+V_2}{2}T-\frac{a(T)}{2}+\frac{V_1\xi_2-V_2\xi_1}{V_1V_2}+\mathrm{i}\left(\frac{V_1+V_2}{2V_1V_2}-\frac{b(T)}{2}\right) \qquad (3.2b)$$

where

$$a(T)=-\frac{(V_1-V_2)^2}{V_1V_2}\frac{T}{b(T)} \qquad (3.3a)$$

$$b(T)=\frac{1}{\sqrt{2}}\Bigg[-\left[(V_1-V_2)^2T^2-\left(\frac{V_1+V_2}{V_1V_2}\right)^2+\frac{4V_{12}}{V_1V_2}\right] +\left\{\left[(V_1-V_2)^2T^2-\left(\frac{V_1+V_2}{V_1V_2}\right)^2+\frac{4V_{12}}{V_1V_2}\right]^2 +4\frac{(V_1-V_2)^4}{(V_1V_2)^2}T^2\right\}^{1/2}\Bigg]^{1/2} \qquad (3.3b)$$

$$T=t+\frac{\xi_1-\xi_2}{V_1-V_2}. \qquad (3.3c)$$

In the following discussion we put $\xi_1=\xi_2=0$ without loss of generality, so that $T=t$.

For $t\to\pm\infty$, $a(t)$ and $b(t)$ behave like

$$a(t)=(V_1-V_2)t+\mathrm{O}(t^{-1}) \qquad (3.4a)$$

$$b(t)=\frac{V_2-V_1}{V_1V_2}+\mathrm{O}(t^{-2}). \qquad (3.4b)$$

Substituting (3.4a) and (3.4b) into (3.2a) and (3.2b), we find

$$x_1(t)=V_1t+\mathrm{i}\frac{1}{V_1}+\mathrm{O}(t^{-1}) \qquad (3.5a)$$

$$x_2(t)=V_2t+\mathrm{i}\frac{1}{V_2}+\mathrm{O}(t^{-1}). \qquad (3.5b)$$

The above results are also derived directly from the asymptotic expressions of $u(x, t)$ for $t\to\pm\infty$ (see I). If we regard solitons as stable particles located at the position of the poles we may introduce the time t_c at which two solitons would collide. It is natural to define it such that the distance between two poles becomes a minimum. In the present case, the distance $l(t)$ between two poles at the time t is given by

$$l(t)=|x_1(t)-x_2(t)|=[a^2(t)+b^2(t)]^{1/2}. \qquad (3.6)$$

It is easy to see from (3.3a)–(3.3c) that $l(t)$ becomes a minimum when $t=0$. Therefore it is interesting to elucidate the behaviour of poles near $t=0$.

First of all, we define the following quantity D which will be seen later to characterise the nature of the interaction between two poles:

$$D = \frac{4V_{12}}{V_1 V_2} - \left(\frac{V_1 + V_2}{V_1 V_2}\right)^2$$
$$= \frac{1}{V_1^2}\frac{(s+1)^2}{s^2}\left(\frac{4s}{(s-1)^2} - 1\right) \tag{3.7}$$

where

$$s = V_2/V_1. \tag{3.8}$$

Note that $V_1^2 D$ decreases monotonically from ∞ to -1 as s increases from 1 to ∞ (see figure 1).

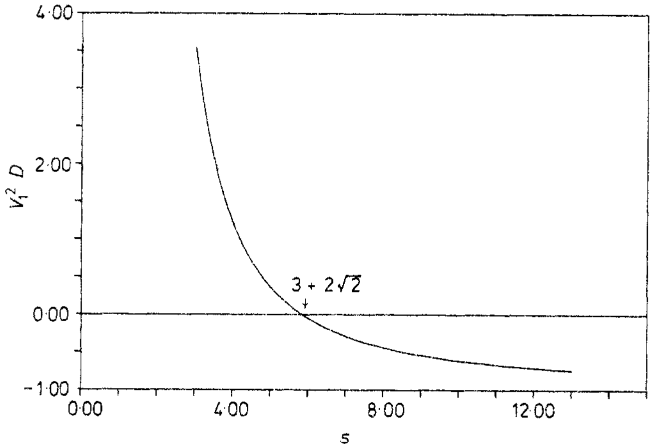

Figure 1. Plot of $V_1^2 D$ as a function of $s(=V_2/V_1)$.

Now we investigate the behaviour of $x_1(t)$ and $x_2(t)$ for small t in three cases:

case A

$$D > 0\ (1 < s < 3 + 2\sqrt{2}) \tag{3.9a}$$

case B

$$D = 0\ (s = 3 + 2\sqrt{2}) \tag{3.9b}$$

case C

$$D < 0\ (s > 3 + 2\sqrt{2}). \tag{3.9c}$$

In the following analysis we describe the situation near $t = 0$ as we want to know the detailed process of the collisions of two solitons.

Case A.

$$a(t) = -\sqrt{D}\frac{t}{|t|}[1 + \mathrm{O}(t^2)] \tag{3.10a}$$

$$b(t) = \frac{1}{\sqrt{D}} |t| [1 + O(t^2)] \tag{3.10b}$$

$$x_1(t) = -\frac{1}{2}\sqrt{D}\frac{t}{|t|} + \mathrm{i}\frac{V_1 + V_2}{V_1 V_2} + O(t) \tag{3.11a}$$

$$x_2(t) = \frac{1}{2}\sqrt{D}\frac{t}{|t|} + \mathrm{i}\frac{V_1 + V_2}{V_1 V_2} + O(t). \tag{3.11b}$$

We see from these expressions that $x_1(t)$ and $x_2(t)$, especially Re $x_1(t)$ and Re $x_2(t)$, are discontinuous at $t = 0$. The minimum distance between two poles is given by

$$l(t_c) = l(0) = D. \tag{3.12}$$

Figure 2 shows a two-soliton solution for various values of time. The parameters are given by

$$V_1 = 1{\cdot}0 \qquad V_2 = 5{\cdot}6 \qquad \xi_1 = \xi_2 = 0 \qquad D = 0{\cdot}0814$$
$$l(t_c) = l(0) = 0{\cdot}285. \tag{3.13}$$

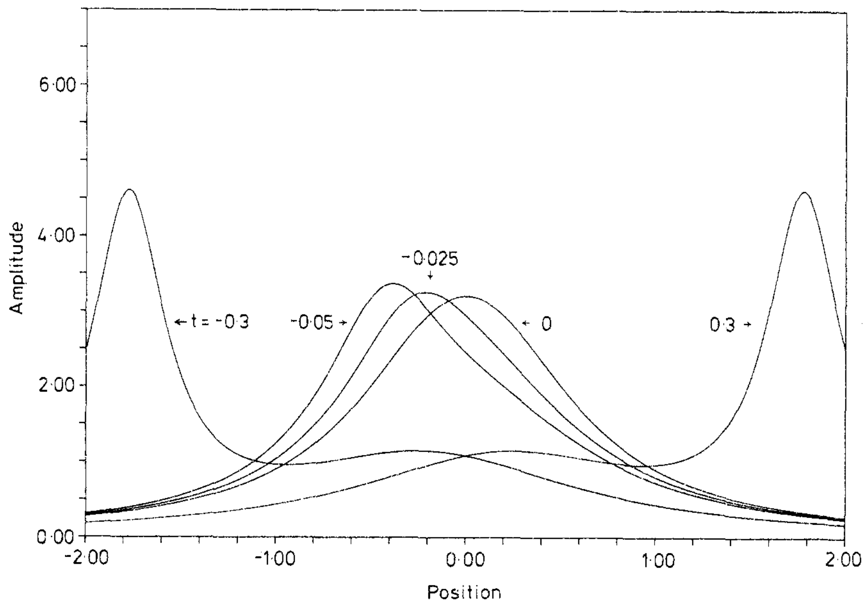

Figure 2. Plot of a two-soliton solution with $V_1 = 1{\cdot}0$, $V_2 = 5{\cdot}6$, $\xi_1 = \xi_2 = 0$ for various values of time.

Figure 3 represents the motions of two poles for $-0{\cdot}3 \leq t \leq 0{\cdot}3$. In figure 4 we plot Re x_1 and Re x_2 as functions of t. We see from these figures that for $D > 0$ two poles interchange their velocities at the instant of the collision ($t = 0$), which reflects the discontinuities of Re x_1 and Re x_2 at $t = 0$ (see (3.11*a*) and (3.11*b*)).

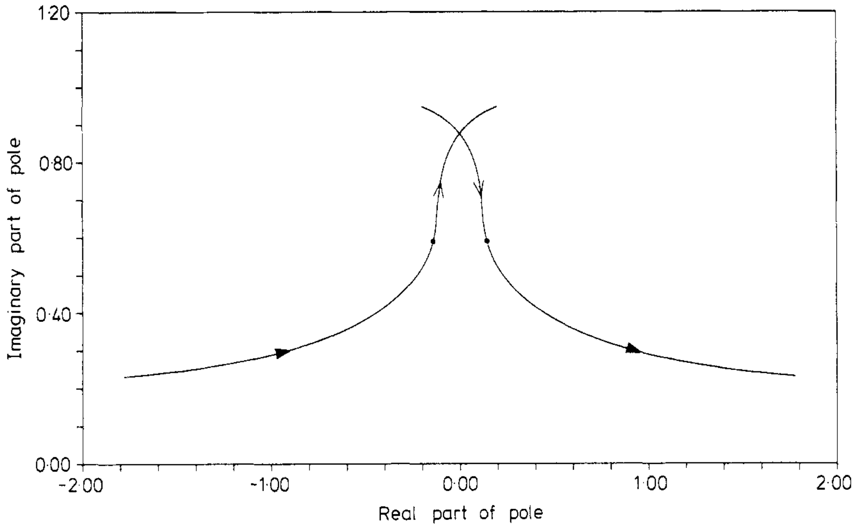

Figure 3. Plot of the motions of two poles for $-0\cdot3 \leqslant t \leqslant 0\cdot3$. The arrow indicates the direction of the motion of the pole and two dots show the positions of two poles at $t=0$. —→—: motion of $x_1(t)$, —➤—: motion of $x_2(t)$.

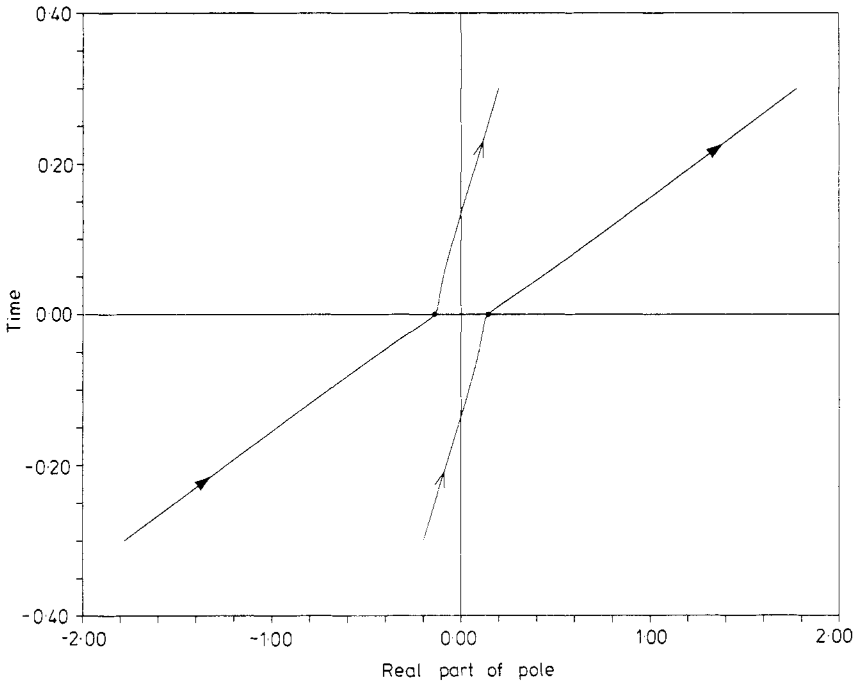

Figure 4. Plot of Re $x_1(t)$ and Re $x_2(t)$ as functions of t.

Case B.

$$a(t) = -\frac{2t}{|t|^{1/2}}[1+O(|t|^{3/2})] \tag{3.14a}$$

$$b(t) = 2|t|^{1/2}[1+O(|t|^{3/2})] \tag{3.14b}$$

$$x_1(t) = -\frac{t}{|t|^{1/2}} + \mathrm{i}\left(\frac{0\cdot586}{V_1} + |t|^{1/2}\right) + \mathrm{O}(t), \tag{3.15a}$$

$$x_2(t) = -\frac{t}{|t|^{1/2}} + \mathrm{i}\left(\frac{0\cdot586}{V_1} - |t|^{1/2}\right) + \mathrm{O}(t). \tag{3.15b}$$

Figures 5, 6 and 7 show various plots corresponding to those of case *A*. The parameters are given by

$$V_1 = 1\cdot0 \qquad V_2 = 3 + 2\sqrt{2} = 5\cdot83 \qquad \xi_1 = \xi_2 = 0$$

$$D = 0 \qquad l(t_c) = l(0) = 0. \tag{3.16}$$

We see from (3.15*a*) and (3.15*b*) that two poles unite at $t = 0$, i.e. $x_1(0) = x_2(0)$. This is shown by a dot in figure 6. Although the trajectories of two poles are continuous, they have a cusp at $t = 0$ (see figure 6).

Case C.

$$a(t) = -\frac{(V_1 - V_2)^2}{V_1 V_2}\frac{t}{\sqrt{-D}}[1 + \mathrm{O}(t^2)] \tag{3.17a}$$

$$b(t) = \sqrt{-D}[1 + \mathrm{O}(t^2)] \tag{3.17b}$$

$$x_1(t) = \left(\frac{V_1 + V_2}{2} - \frac{(V_1 - V_2)^2}{2V_1 V_2}\frac{1}{\sqrt{-D}}\right)t + \mathrm{i}\left(\frac{V_1 + V_2}{2V_1 V_2} + \frac{\sqrt{-D}}{2}\right) + \mathrm{O}(t^2) \tag{3.18a}$$

$$x_2(t) = \left(\frac{V_1 + V_2}{2} + \frac{(V_1 - V_2)^2}{2V_1 V_2}\frac{1}{\sqrt{-D}}\right)t + \mathrm{i}\left(\frac{V_1 + V_2}{2V_1 V_2} - \frac{\sqrt{-D}}{2}\right) + \mathrm{O}(t^2) \tag{3.18b}$$

$$l(t_c) = l(0) = -D. \tag{3.19}$$

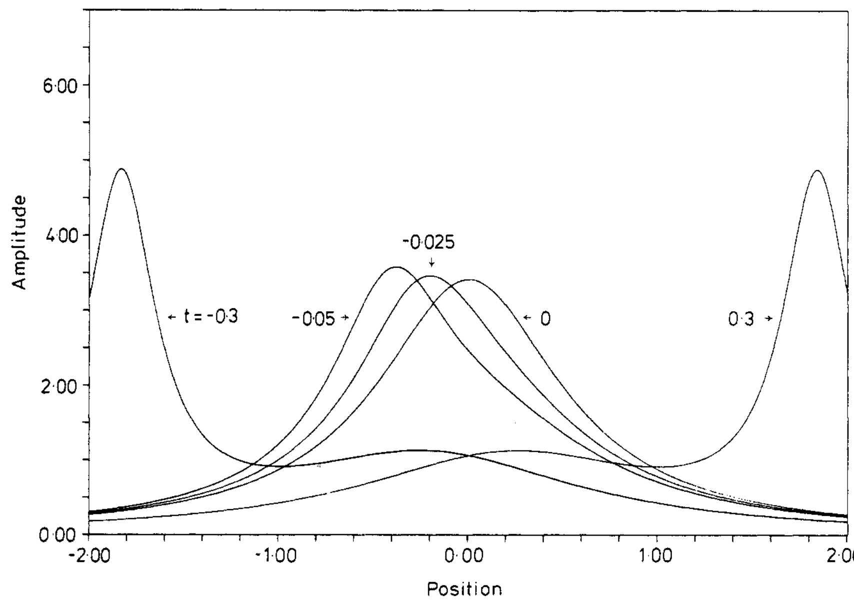

Figure 5. Plot is the same as figure 2 except $V_2 = 3 + 2\sqrt{2}$.

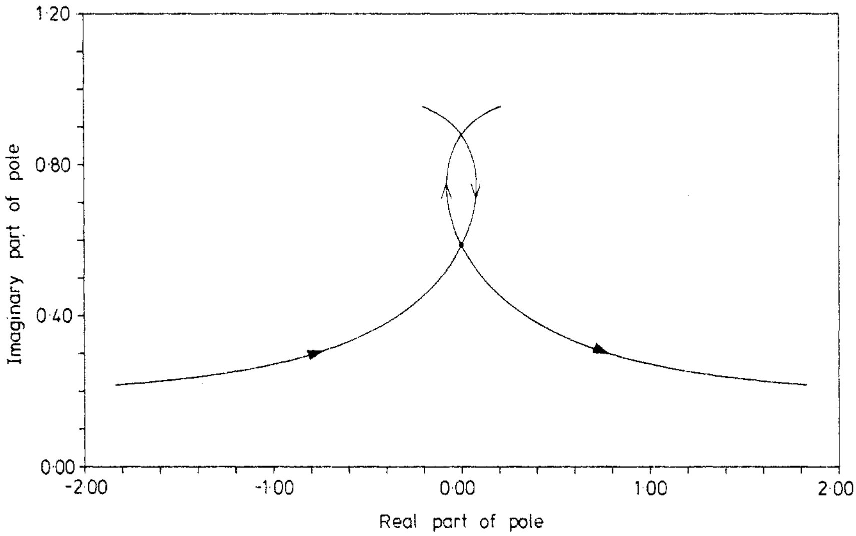

Figure 6. Plot is the same as figure 3 except $V_2 = 3 + 2\sqrt{2}$.

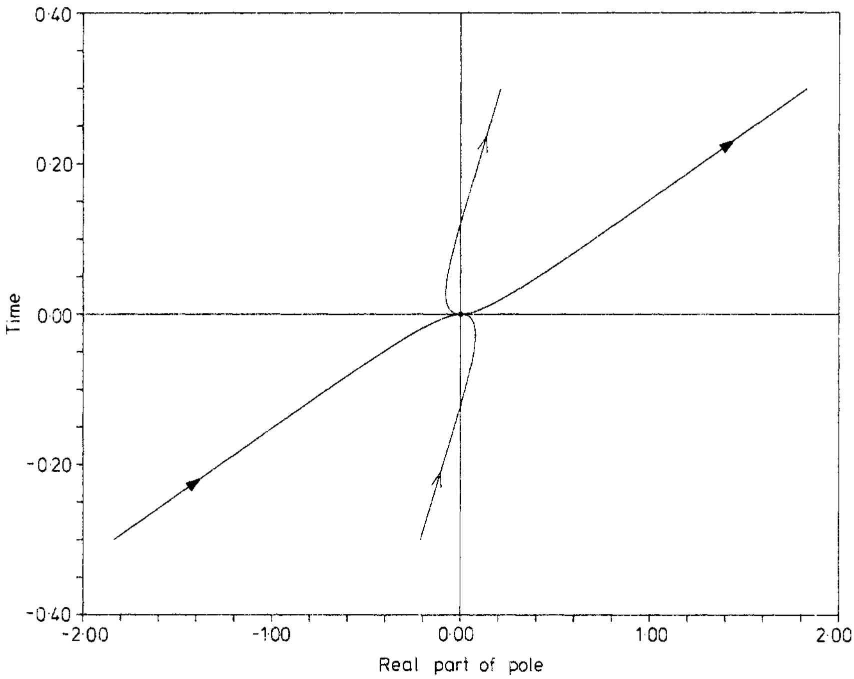

Figure 7. Plot is the same as figure 4 except $V_2 = 3 + 2\sqrt{2}$.

Figures 8, 9 and 10 show various plots corresponding to those of case *A*. The parameters are given by

$$V_1 = 1{\cdot}0 \qquad V_2 = 6{\cdot}0 \qquad \xi_1 = \xi_2 = 0 \qquad D = -0{\cdot}0544$$
$$l(t_c) = l(0) = 0{\cdot}233. \tag{3.20}$$

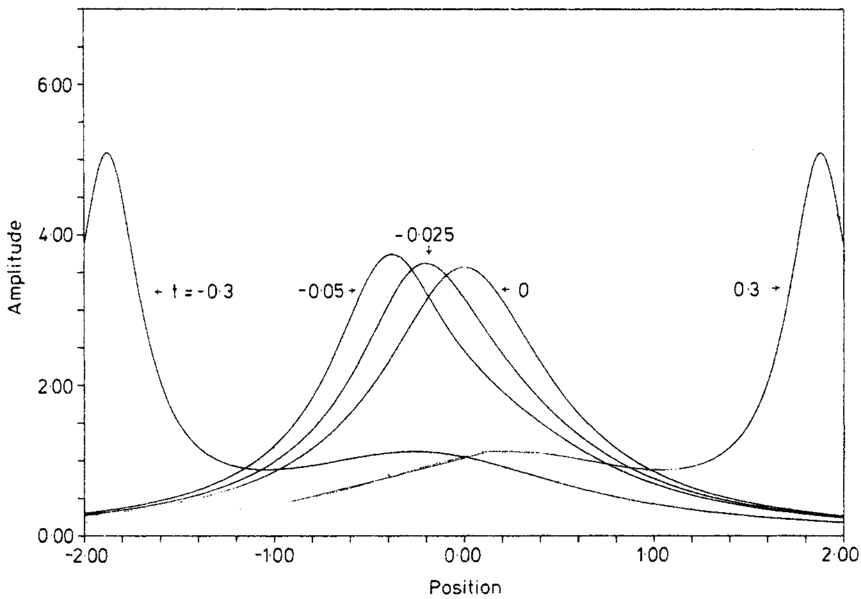

Figure 8. Plot is the same as figure 2 except $V_2 = 6 \cdot 0$.

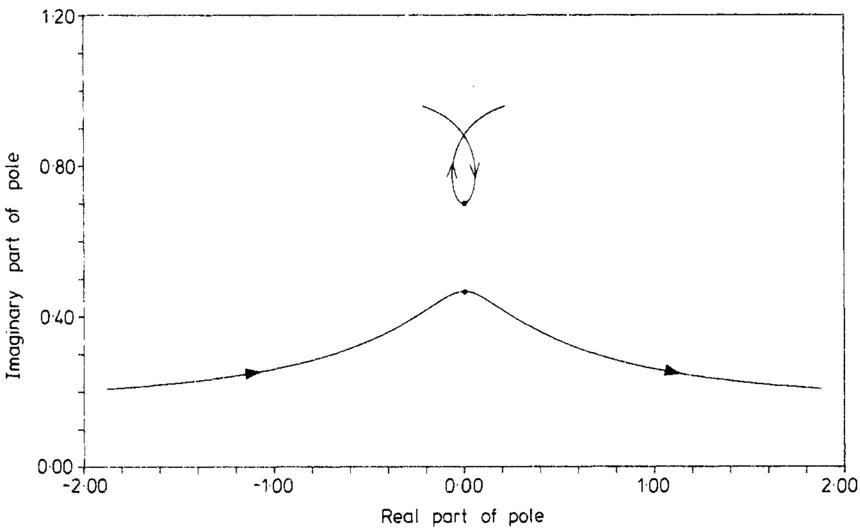

Figure 9. Plot is the same as figure 3 except $V_2 = 6 \cdot 0$.

From these figures we find that the trajectories of two poles are continuous and have no cusps for all times and this point makes a clear difference between case A and case C. Note also that $\mathrm{d}/\mathrm{d}t \operatorname{Re} x_1(t)$ which represents the speed of the smaller soliton becomes zero at two times (from figure 10 these times are seen to be about $\pm 0 \cdot 03$) and has a minimum value at $t = 0$, which is given by

$$\min \frac{\mathrm{d}}{\mathrm{d}t} \operatorname{Re} x_1(t) = \frac{\mathrm{d}}{\mathrm{d}t} \operatorname{Re} x_1(t)\Big|_{t=0} = \frac{V_1 + V_2}{2} - \frac{(V_1 - V_2)^2}{2 V_1 V_2} \frac{1}{\sqrt{-D}} \tag{3.21}$$

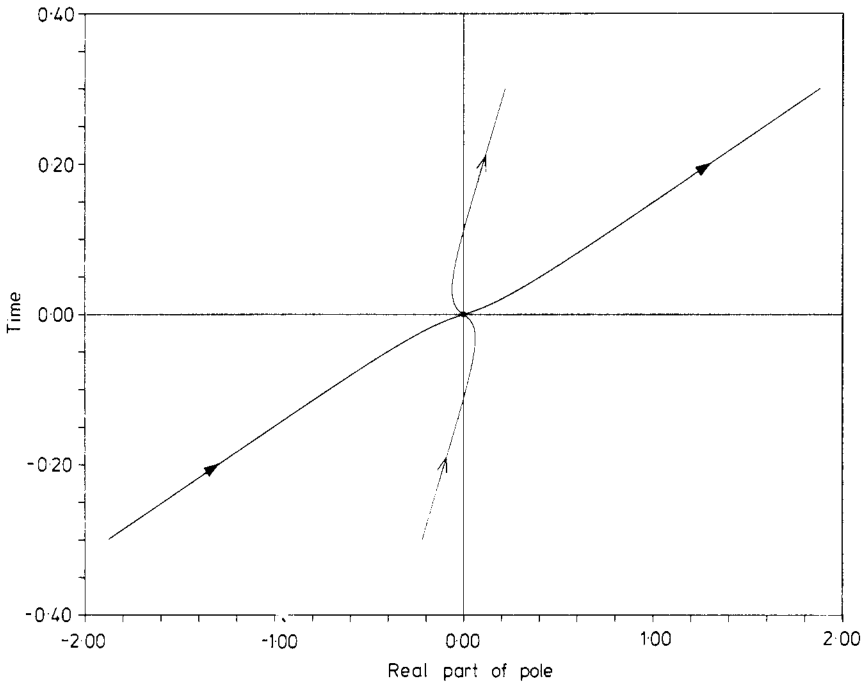

Figure 10. Plot is the same as figure 4 except $V_2 = 6{\cdot}0$.

where we have used (3.18*a*). This minimum value is $-5{\cdot}4$ in the present example. On the other hand, $\mathrm{d}/\mathrm{d}t\ \mathrm{Re}\ x_2(t)$, which represents the speed of the larger soliton, never becomes zero and has the following maximum value at $t = 0$:

$$\max \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{Re}\ x_2(t) = \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{Re}\ x_2(t)|_{t=0} = \frac{V_1 + V_2}{2} + \frac{(V_1 - V_2)^2}{2V_1 V_2} \frac{1}{\sqrt{-D}} \tag{3.22}$$

where we have used (3.18*b*). In the present example (see (3.20)) this maximum value is 12·4.

It should be pointed out here that although the process of the interaction of two solitons is very different for the three cases, the profiles of the two-soliton solutions obtained are similar to one another, i.e. they have only one peak at the instant of the collision (see figures 2, 5 and 8). The profile of the two-soliton solution at $t = 0$ is found from (2.11) with $\xi_1 = \xi_2 = 0$ as

$$u(x, 0) = \frac{V_1 V_2 (V_1 + V_2) x^2 + (V_1 + V_2) V_{12}}{(V_1 V_2 x^2 - V_{12})^2 + (V_1 + V_2)^2 x^2}. \tag{3.23}$$

This function has different profiles depending on the values of a parameter $s = V_2/V_1$.

(*a*)

$$1 < s < \frac{5 + \sqrt{21}}{2} = 4{\cdot}79$$

$$\max u(x,0) = u(\pm x_0, 0)$$
$$= V_1 \frac{(s-1)^2}{s+1} \frac{s}{(-s^2+6s-1)^{1/2}[2s^{1/2}-(-s^2+6s-1)^{1/2}]} \tag{3.24}$$

where

$$x_0 = \frac{1}{V_1}\frac{1}{\sqrt{s}}\frac{s+1}{s-1}\left[\left(\frac{-s^2+6s-1}{s}\right)^{1/2}-1\right]^{1/2}. \tag{3.25}$$

(*b*)

$$s > \frac{5+\sqrt{21}}{2}$$

$$\max u(x,0) = u(0,0) = V_1 \frac{(s-1)^2}{s+1}. \tag{3.26}$$

It is interesting to note that for $1<s<3+2\sqrt{2}$ the positions of maximum values of $u(x,0)$ become $\pm x_0$ (for $1<s<(5+\sqrt{21})/2$) and 0 (for $(5+\sqrt{21})/2<s<3+2\sqrt{2}$). These positions do not coincide with the positions of maximum values of two solitons given by

$$\mathrm{Re}\, x_1(-0) = \frac{\sqrt{D}}{2} = \frac{1}{V_1}\frac{1}{\sqrt{s}}\frac{s+1}{s-1}\left(1-\frac{(s-1)^2}{4s}\right)^{1/2} \tag{3.27a}$$

$$\mathrm{Re}\, x_2(-0) = -\frac{\sqrt{D}}{2} \tag{3.27b}$$

where

$$x_0 < \mathrm{Re}\, x_1(-0) \qquad \text{for } s>1. \tag{3.28}$$

However, the process of the interaction of two solitons for $1<s<(5+\sqrt{21})/2$ is not substantially different from that for $1<s<3+2\sqrt{2}$ as has already been shown in case *A*. The circumstances mentioned above are explained in figures 11, 12 and 13, where various plots corresponding to those of case *A* are given. The parameters are

$$V_1 = 1{\cdot}0 \qquad V_2 = 4{\cdot}6 \qquad \xi_1 = \xi_2 = 0 \qquad D = 0{\cdot}622$$
$$l(t_c) = l(0) = 0{\cdot}789. \tag{3.29}$$

It follows from (3.24), (3.25), (3.27*a*) and (3.27*b*) that

$$x_0 = 0{\cdot}215 \qquad \max u(x,0) = u(\pm x_0, 0) = 2{\cdot}33$$
$$\mathrm{Re}\, x_1(-0) = -\mathrm{Re}\, x_2(-0) = 0{\cdot}394. \tag{3.30}$$

To summarise, the nature of the interaction of two solitons can be divided into two classes depending on the initial amplitudes of two solitons as follows.

For $V_1 < V_2 < (3+2\sqrt{2})V_1$, as the time goes from $-\infty$ to ∞, the amplitude of the larger soliton decreases from V_2 to V_1 while the amplitude of the smaller one increases from V_1 to V_2, in which case two solitons interchange their velocities at the instant of the collision without passing through each other (see figures 3 and 4).

For $V_2 > (3+2\sqrt{2})V_1$, on the other hand, the larger soliton first absorbs the smaller one and then emits the smaller one backward, which is clear from the fact that the speed

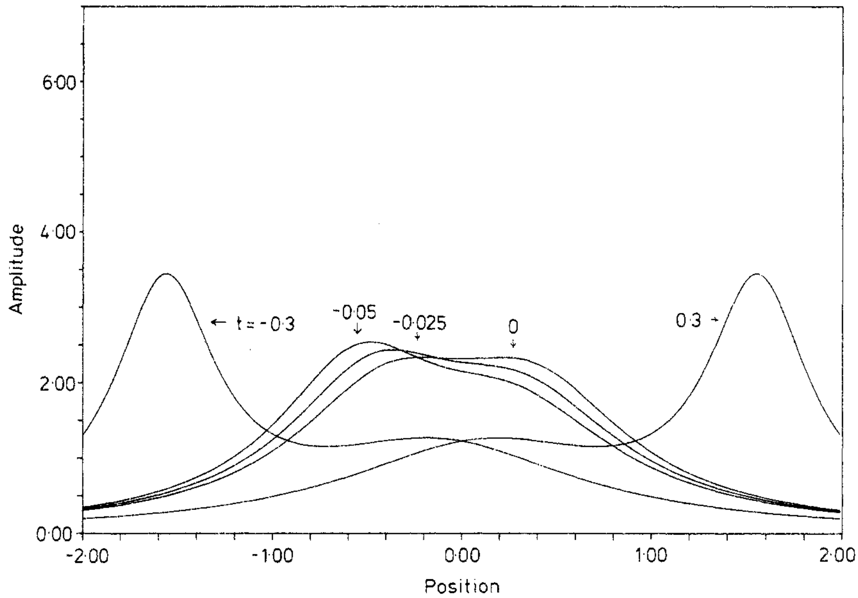

Figure 11. Plot is the same as figure 2 except $V_2 = 4{\cdot}6$.

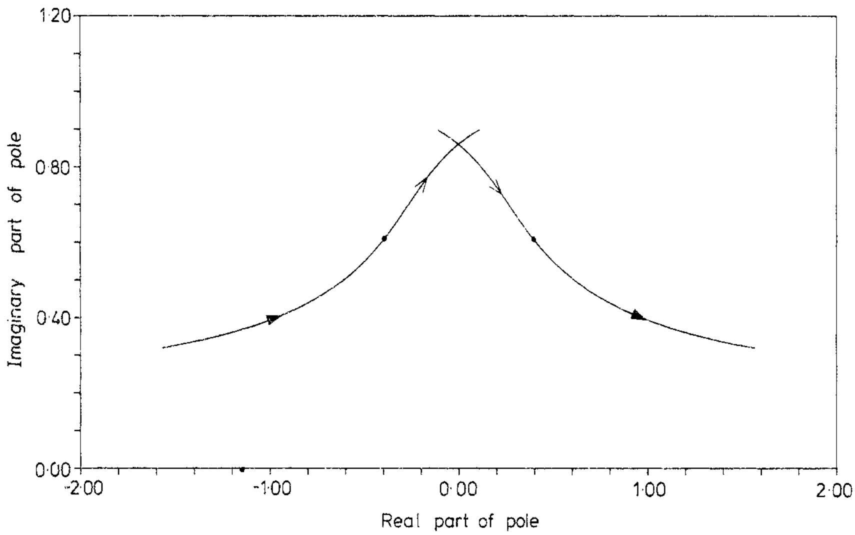

Figure 12. Plot is the same as figure 3 except $V_2 = 4{\cdot}6$.

of the smaller soliton becomes negative for some range of time (see figure 10, in which example this range is $-0{\cdot}03 \leq t \leq 0{\cdot}03$). Thus, in this case two solitons pass through each other.

4. Linearised Benjamin–Ono equation

As has already been mentioned, the behaviour of the tail is well known thanks to the inverse scattering method in the case of the KdV equation. Since this method cannot be

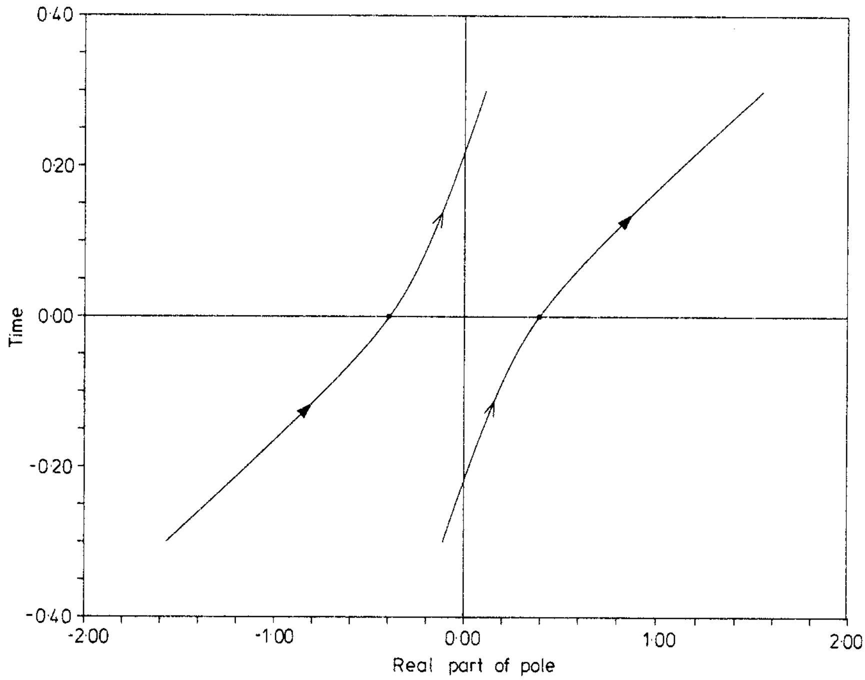

Figure 13. Plot is the same as figure 4 except $V_2 = 4{\cdot}6$.

applied to the B–O equation as yet, we must resort to other methods to study the structure of the tail. One of them may be to study the solution of the linearised B–O equation,

$$\frac{\partial u(x,t)}{\partial t} + H\left[\frac{\partial^2 u(x,t)}{\partial x^2}\right] = 0; \tag{4.1}$$

because the amplitude of the tail is small it may be described by the linearised B–O equation (4.1) in the first approximation.

In this section, we consider the initial-value problem of the equation (4.1) and derive the asymptotic expression of $u(x, t)$ for both large x and t.

Now the solution of the equation (4.1) can be represented by the Fourier integral as follows:

$$u(x,t) = \int_{-\infty}^{\infty} v(k) \exp\{i[kx - \omega(k)t]\}\, dk. \tag{4.2}$$

If we substitute (4.2) into (4.1), we can determine the functional form of $\omega(k)$ as

$$\omega(k) = -k|k| \tag{4.3}$$

where we have used the formula

$$P\int_{-\infty}^{\infty} \frac{e^{ikx}}{x}\, dx = i\pi\frac{|k|}{k}. \tag{4.4}$$

The unknown function $v(k)$ is determined from the initial value of $u(x, t)$ as follows:

$$v(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,0)\, e^{-ikx}\, dx. \tag{4.5}$$

On introducing (4.3) and (4.5) into (4.2), we get the general solution of the equation (4.1) as

$$u(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\{\mathrm{i}[k(x-y)+k|k|t]\}u(y,0)\,\mathrm{d}k\,\mathrm{d}y$$

$$=\frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{\infty}K\left(\frac{x-y}{2\sqrt{t}}\right)u(y,0)\,\mathrm{d}y \tag{4.6}$$

where

$$K(x)=[\tfrac{1}{2}-C(\sqrt{2/\pi}\,x)]\cos(x^2)+[\tfrac{1}{2}-S(\sqrt{2/\pi}\,x)]\sin(x^2) \tag{4.7}$$

$$C(x)=\int_0^x\cos\left(\frac{\pi}{2}t^2\right)\mathrm{d}t \qquad S(x)=\int_0^x\sin\left(\frac{\pi}{2}t^2\right)\mathrm{d}t. \tag{4.8}$$

The function $K(x)$ has the following asymptotic representation:

$$K(x)\sim\sqrt{2/\pi}\sum_{n=0}^{\infty}\frac{(-1)^n(4n+1)!!}{2^{2n+2}x^{4n+3}} \qquad (x\to+\infty) \tag{4.9a}$$

$$K(x)\sim\cos(x^2)+\sin(x^2)-\sqrt{2/\pi}\sum_{n=0}^{\infty}\frac{(-1)^n(4n+1)!!}{2^{2n+2}x^{4n+3}} \qquad (x\to-\infty). \tag{4.9b}$$

Now let us examine the asymptotic form of (4.6) at large values of t on the assumption that the initial perturbation vanishes rapidly when $|x|\to\infty$. Then, we can express $K[(x-y)/2\sqrt{t}]$ in the form of a power series in y. It follows readily from (4.6) that

$$u(x,t)=\frac{1}{\sqrt{2\pi t}}\sum_{n=0}^{\infty}\frac{(-1)^n p_n}{n!}\left(\frac{1}{2\sqrt{t}}\right)^n K^{(n)}\left(\frac{x}{2\sqrt{t}}\right) \tag{4.10}$$

where

$$p_n=\int_{-\infty}^{\infty}y^n u(y,0)\,\mathrm{d}y \tag{4.11}$$

$$K^{(n)}(x)=\frac{\mathrm{d}^n}{\mathrm{d}x^n}K(x). \tag{4.12}$$

Substituting into (4.10) the asymptotic expression for the derivatives of $K(x)$

$$K^{(n)}(x)\sim\sqrt{2/\pi}\,\frac{(-1)^n}{8}\frac{(n+2)!}{x^{n+3}} \qquad (x\to+\infty) \tag{4.13a}$$

$$K^{(n)}(x)\sim\sqrt{2}\,(2x)^n\cos\left(x^2-\frac{\pi}{4}+\frac{n\pi}{2}\right) \qquad (x\to-\infty) \tag{4.13b}$$

and using the Fourier transform of $u(x,0)$

$$v(k)=\frac{1}{2\pi}\sum_{n=0}^{\infty}p_n\frac{(-\mathrm{i}k)^n}{n!} \tag{4.14}$$

we get as $x/\sqrt{t}\to-\infty$

$$u(x,t)\sim 2\sqrt{\pi/t}\,\mathrm{Re}\left\{v(x/2t)\exp\left[\mathrm{i}\left(\frac{x^2}{4t}-\frac{\pi}{4}\right)\right]\right\}. \tag{4.15}$$

As $x/\sqrt{t}\to\infty$, on the other hand, we have

$$u(x,t)\sim\frac{1}{\pi\sqrt{t}}\frac{2}{(x/\sqrt{t})^3}p_0 \qquad (p_0\neq 0). \tag{4.16}$$

For $|x|/\sqrt{t}\leqslant 1$ and large values of t, we can approximate the solution by the first non-vanishing term in (4.10). Thus, we get

$$u(x,t)\sim\frac{1}{\sqrt{2\pi t}}K\left(\frac{x}{2\sqrt{t}}\right)p_0 \qquad (p_0\neq 0). \tag{4.17}$$

Finally we take, as an example, the δ function initial condition:

$$u(x,0)=\delta(x). \tag{4.18}$$

The solution for $t>0$ is obtained from (4.6) as

$$u(x,t)=\frac{1}{\sqrt{2\pi t}}K\left(\frac{x}{2\sqrt{t}}\right). \tag{4.19}$$

For $|x/\sqrt{t}|\to\infty$, we have, from (4.9a) and (4.9b),

$$u(x,t)\sim\frac{1}{\pi\sqrt{t}}\frac{2}{(x/\sqrt{t})^3} \qquad \text{for } x/\sqrt{t}\to+\infty \tag{4.20}$$

$$u(x,t)\sim\frac{1}{\sqrt{\pi t}}\cos\left(\frac{x^2}{4t}-\frac{\pi}{4}\right) \qquad \text{for } x/\sqrt{t}\to-\infty. \tag{4.21}$$

These asymptotic behaviours are special cases of (4.15) and (4.16).

5. Concluding remarks

In the preceding sections, we have studied the detailed process for the interaction of two solitons. Similar analysis may be carried out for the interaction of $N(\geqslant 3)$ solitons by using the explicit N-soliton solution given by (2.6)–(2.8), which will be done elsewhere.

Acknowledgments

The author wishes to express his cordial thanks to Professor T Tatsumi and Dr T Kawahara for encouragement and to Mr M Tanaka for assistance with numerical work.

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